

AN EGG-YOLK PRINCIPLE AND EXPONENTIAL INTEGRABILITY FOR QUASIREGULAR MAPPINGS

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ABSTRACT. Quasiregular mappings $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ are a natural generalization of analytic functions from complex analysis and provide a theory which is rich with new phenomena. In this paper we extend a well-known result of A. Chang and D. Marshall on exponential integrability of analytic functions in the disk, to the case of quasiregular mappings defined in the unit ball of \mathbb{R}^n . To this end, we first establish an “egg-yolk” principle for such maps, which extends a recent result of the first author. Our work leaves open an interesting problem regarding n -harmonic functions.

1. INTRODUCTION

We will denote an n -dimensional ball with center a and radius r by $\mathbb{B}^n(a, r)$. The unit ball is \mathbb{B}^n . Sometimes the notation $r\mathbb{B}^n$ for $\mathbb{B}^n(0, r)$ is used. Similarly, the notations $\mathbb{S}^{n-1}(a, r)$ and \mathbb{S}^{n-1} for the corresponding $(n-1)$ -spheres will be used, respectively. The s -dimensional Hausdorff measure will be denoted by \mathcal{H}_s . The volume of \mathbb{B}^n is denoted by α_n , and the $(n-1)$ -measure of \mathbb{S}^{n-1} by ω_{n-1} .

A mapping $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called **quasiregular (qr)** if it belongs to the Sobolev class $W_{loc}^{1,n}(\Omega, \mathbb{R}^n)$, and, for some $K \geq 1$, it satisfies the distortion inequality

$$\|Df(x)\|^n \leq KJ(x, f)$$

for almost every $x \in \Omega$, where $\|Df(x)\|$ is the operator norm of the matrix derivative $Df(x) = \left(\frac{\partial f_i}{\partial x_j}\right)_{i,j=1}^n$, which is well-defined for almost every $x \in \mathbb{R}^n$, and $J(x, f)$ is the Jacobian determinant of f at x , i.e., $J(x, f) = \det Df(x)$. It is well-known that quasiregular mappings are continuous and almost everywhere differentiable, and, when non-constant, they are open and discrete. Also when $n = 2$ and $K = 1$ they are analytic functions. They provide a fruitful generalization of classical function theory to higher (real) dimensional spaces. We refer to [Res89] and [Ric93] for the basic theory of quasiregular mappings. The theory of these mappings is often referred to, in colorful language, as the **quasiworld**.

The purpose of this paper is twofold. We extend the exponential integrability result of [CM85] to the quasiworld. But, to do this, we also need to extend an “egg-yolk principle for

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the inverse map” conjectured by D. Marshall in [Mar89], which has been shown to hold in the classical case in [PC].

1.1. Exponential integrability. The following result is proved in [CM85].

Theorem A (Chang-Marshall [CM85]). *There is a universal constant $C < \infty$ so that if f is analytic in \mathbb{D} , $f(0) = 0$, and*

$$(1.1) \quad \int_{\mathbb{D}} |f'(z)|^2 dA(z)/\pi \leq 1,$$

then

$$\int_0^{2\pi} \exp(|f^*(e^{i\theta})|^2) d\theta \leq C.$$

where f^ is the trace of f on $\partial\mathbb{D}$, i.e., $f^*(\zeta) = \lim_{t \uparrow 1} f(t\zeta)$ for \mathcal{H}_1 -a.e. $\zeta \in \partial\mathbb{D}$.*

This result is moreover “sharp”. Indeed, even though for any given $\beta > 0$ and any analytic function f on \mathbb{D} , satisfying $f(0) = 0$ and (1.1), the integral

$$\int_0^{2\pi} \exp(\beta |f^*(e^{i\theta})|^2) d\theta$$

is finite, there is a family of functions, the Beurling functions

$$B_a(z) = \left(\log \frac{1}{1-az} \right) \left(\log \frac{1}{1-a^2} \right)^{-\frac{1}{2}} \quad 0 < a < 1$$

that are analytic in \mathbb{D} , satisfy $B_a(0) = 0$ and (1.1), with the property that for any given $\alpha > 1$, one can choose a so that the integral

$$\int_0^{2\pi} \exp(\alpha |B_a(e^{i\theta})|^2) d\theta$$

is as large as desired.

In this paper we extend the Chang-Marshall result to quasiregular mappings.

Theorem 1.1. *There exists a constant $C = C(n, K) < \infty$ so that if $f : \mathbb{B}^n \rightarrow \mathbb{R}^n$, $n \geq 2$, is a K -quasiregular mapping with $f(0) = 0$ and*

$$(1.2) \quad \int_{\mathbb{B}^n} J(x, f) dx \leq \alpha_n,$$

then

$$\int_{\mathbb{S}^{n-1}} \exp \left((n-1) \left(\frac{n}{2K} \right)^{\frac{1}{n-1}} |f^*(\zeta)|^{\frac{n}{n-1}} \right) d\mathcal{H}_{n-1}(\zeta) \leq C,$$

where f^ is the trace of f on \mathbb{S}^{n-1} , i.e., $f^*(\zeta) = \lim_{t \uparrow 1} f(t\zeta)$ for \mathcal{H}_{n-1} -a.e. $\zeta \in \mathbb{S}^{n-1}$.*

The trace f^* in Theorem 1.1 is well-defined, since a quasiregular mapping $f : \mathbb{B}^n \rightarrow \mathbb{R}^n$ satisfying (1.2) has radial limits at almost every $\theta \in \mathbb{S}^{n-1}$, see [Ric93], VII Theorem 2.7.

For a mapping satisfying the assumptions of Theorem 1.1,

$$\int_{\mathbb{S}^{n-1}} \exp \left(\beta |f^\star(\zeta)|^{\frac{n}{n-1}} \right) d\mathcal{H}_{n-1}(\zeta) < \infty$$

for every $\beta > 0$. This is a consequence of Theorem 1.5 as will be shown at the end of Section 5.

Theorem 1.1 is sharp for $n = 2$, in the sense that for any $K \geq 1$ the constant K^{-1} cannot be improved on. To see this, first map the unit disk onto the upper half plane by a Möbius transformation, so that $(1, 0)$ is mapped to the origin. Then apply the radial stretching $z \mapsto z|z|^{K-1}$, which is a K -quasiconformal map, and map back to the disk. Finally, apply the Beurling functions B_a . The compositions of these maps, $B_{K,a}$, are K -quasiregular maps satisfying the assumptions of Theorem 1.1, and for each $\beta > K^{-1}$,

$$\sup_{0 < a < 1} \int_0^{2\pi} \exp \left(\beta |B_{K,a}^\star(e^{i\theta})|^2 \right) d\theta = \infty.$$

In dimensions higher than two the situation is different. Indeed, by the Liouville theorem of Gehring and Reshetnyak, see [Res89], Theorem 5.10, 1-quasiregular mappings in dimensions three or higher are Möbius transformations. Moreover, the L^∞ -norm of a Möbius transformation satisfying the assumptions of Theorem 1.1 is bounded by two. We expect that the constant $(n-1) \left(\frac{n}{2K} \right)^{\frac{1}{n-1}}$ is not sharp for any $n \geq 3$ and any $K \geq 1$. In particular, it would be interesting to determine whether the sharp constant stays bounded as n tends to infinity. Spatial maps that are similar to the Beurling functions can be constructed by using cylinder maps (K -quasiconformal maps mapping \mathbb{B}^n onto an infinite cylinder). The best dilatation constant K for cylinder maps is not known, see [GV65], Section 8.

1.2. Further remarks. The Chang-Marshall theorem has the following two corollaries for harmonic and Sobolev functions.

Corollary D. *There is a universal constant $C < \infty$ so that if $u : \mathbb{D} \rightarrow \mathbb{R}$ is harmonic with $u(0) = 0$ and*

$$\int_{\mathbb{D}} |\nabla u(z)|^2 dA(z)/\pi \leq 1,$$

then

$$\int_0^{2\pi} \exp(u^\star(e^{i\theta})^2) d\theta \leq C$$

where u^\star is the trace of u on $\partial\mathbb{D}$, i.e., $u^\star(\zeta) = \lim_{t \uparrow 1} u(t\zeta) = u^\star(\zeta)$ for \mathcal{H}_1 -a.e. $\zeta \in \partial\mathbb{D}$.

Proof. Let \tilde{u} be the harmonic conjugate of u such that $\tilde{u}(0) = 0$. Then $f = u + i\tilde{u}$ satisfies the hypothesis of Theorem C, since $|f'| = |\nabla u|$. So

$$\int_0^{2\pi} \exp(u^\star(e^{i\theta})^2) d\theta \leq \int_0^{2\pi} \exp(u^\star(e^{i\theta})^2 + \tilde{u}^\star(e^{i\theta})^2) d\theta \leq C.$$

□

Corollary E. *There is a universal constant $C < \infty$ so that if $v \in W^{1,2}(\mathbb{D})$ with $\int_{\partial\mathbb{D}} v^*(e^{i\theta}) d\theta = 0$ and*

$$\int_{\mathbb{D}} |\nabla v(z)|^2 dA(z)/\pi \leq 1,$$

then

$$\int_0^{2\pi} \exp(v^*(e^{i\theta})^2) d\theta \leq C$$

where v^ is the Sobolev trace of v on $\partial\mathbb{D}$.*

For the concept of Sobolev trace see [Zie89], pages 189–191.

Proof. Let v^* be the trace of v on the circle $\partial\mathbb{D}$. Solve the Dirichlet problem with these boundary values, to get u harmonic in \mathbb{D} with

$$\int_{\mathbb{D}} |\nabla u|^2 dA/\pi \leq \int_{\mathbb{D}} |\nabla v|^2 dA/\pi \leq 1.$$

Then Corollary D implies $\int_0^{2\pi} \exp(u^*(e^{i\theta})^2) d\theta \leq C$, but $u^* = v^*$. So the same is true for v^* . \square

Remark 1.2. In terms of statements we have:

$$\text{Theorem A} \implies \text{Corollary D} \iff \text{Corollary E}$$

Corollary E could possibly be proved by “Sobolev” methods, see for instance the similar Theorem 3.2.1 of [AH96]. When a seemingly stronger normalization

$$\int_{\frac{1}{2}\mathbb{B}^n} u(x) dx = 0$$

is assumed, the techniques below can be used to prove results like Corollary E in all dimensions, see comments at the end of Section 4.

Remark 1.3. Condition (1.1) says that the Euclidean area of $f(\mathbb{D})$ counting multiplicity is less or equal to π . In [Ess87] it is shown that (1.1) can be replaced by the condition that the area of the set $f(\mathbb{D})$ is less or equal to π , without counting multiplicity.

1.3. Open Questions. In view of Corollary D we ask:

Question 1.4. What is the best constant β for which there exists $C > 0$ so that if $u \in W^{1,n}(\mathbb{B}^n)$, $n \geq 2$, is n -harmonic on \mathbb{B}^n , $u(0) = 0$, and

$$\int_{\mathbb{B}^n} |\nabla u(x)|^n dx \leq \alpha_n,$$

then

$$\int_{\mathbb{S}^{n-1}} \exp\left(\beta |u^*(\zeta)|^{\frac{n}{n-1}}\right) d\mathcal{H}_{n-1}(\zeta) \leq C?$$

1.4. Beurling's estimate. In [Mar89], Don Marshall deduces Theorem A from an estimate of Beurling, Theorem B below. We denote $E_t = \{x \in \mathbb{B}^n : |f(x)| = t\}$, and $F_s^* = \{\theta \in \mathbb{S}^{n-1} : |f(\theta)| > s\}$. The following is an unpublished estimate of A. Beurling which is stated and proved in [Mar89]. Here “Cap” denotes logarithmic capacity.

Theorem B (Beurling). *Suppose f is analytic in a neighborhood of $\overline{\mathbb{D}}$ and suppose that $|f(z)| \leq M$ for $|z| \leq r < 1$, for some $0 < r < 1$. Then, for every $s > M$,*

$$\text{Cap } F_s^* \leq r^{\frac{-1}{2}} \exp \left(-\pi \int_M^s \frac{dt}{|f(E_t)|} \right)$$

where $|f(E_t)|$ denotes the length of $f(E_t)$ counting multiplicity.

We establish a similar estimate in space. For a quasiregular map $f : \mathbb{B}^n \rightarrow \mathbb{R}^n$, $n \geq 2$, we denote the $(n-1)$ -measure of $f(E_t)$ counting multiplicity by $\mathcal{A}_{n-1}f(E_t)$;

$$\mathcal{A}_{n-1}f(E_t) = \int_{\mathbb{S}^{n-1}(0,t)} \text{card } f^{-1}(y) d\mathcal{H}_{n-1}(y).$$

Theorem 1.5. *Let f be a K -quasiregular mapping defined in a neighborhood of $\overline{\mathbb{B}^n}$, $n \geq 2$, and suppose that $|f(x)| \leq M$ for $|x| \leq r < 1$. Then, for every $s > M$,*

$$(1.3) \quad \mathcal{H}_{n-1}(F_s^*) \leq C_1 \exp \left((1-n) \left(\frac{\omega_{n-1}}{2K} \right)^{\frac{1}{n-1}} \int_M^s \frac{dt}{(\mathcal{A}_{n-1}f(E_t))^{\frac{1}{n-1}}} \right),$$

where C_1 depends only on n , K and r .

1.5. An egg-yolk principle for the inverse. In [Mar89], Don Marshall conjectures an egg-yolk principle that would have simplified his argument for passing from Theorem B to Theorem A. This was proved in [PC] by the first author.

Theorem C ([PC]). *There is a universal constant $0 < r_0 < 1$ such that whenever f is analytic on $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ with $f(0) = 0$, and whenever $M > 0$ is such that*

$$\int_{\{z \in \mathbb{D} : |f(z)| < M\}} |f'(z)|^2 dA(z) < \pi M^2,$$

then we have that $|z| < r_0$ implies $|f(z)| < M$.

Here we prove that Theorem C extends to quasiregular maps, and this will allow us to deduce Theorem 1.1 from Theorem 1.5.

Theorem 1.6. *Given $n \geq 2$ and $K \geq 1$, there exists a constant $0 < r_0(n, K) < 1$, so that whenever $f : \mathbb{B}^n \rightarrow \mathbb{R}^n$ is a K -quasiregular mapping with $f(0) = 0$ and whenever $M > 0$ is such that*

$$(1.4) \quad \int_{\{x \in \mathbb{B}^n : |f(x)| < M\}} J(x, f) dx < \alpha_n M^n,$$

then we have that $|x| < r_0$ implies $|f(x)| < M$.

Theorem 1.6 is equivalent to the following.

Corollary 1.7. *For $n \geq 2$ and $K \geq 1$, there exists a constant $0 < r_0(n, K) < 1$ so that if $f : \mathbb{B}^n \rightarrow \mathbb{R}^n$ is a K -quasiregular mapping with $f(0) = 0$, then $0 \leq M < \max_{|x| \leq r_0} |f(x)|$ implies*

$$\int_{\{x \in \mathbb{B}^n : |f(x)| < M\}} J(x, f) dx \geq \alpha_n M^n.$$

Theorem 1.6 no longer holds true if instead of (1.4) it is assumed that $\mathbb{B}^n \setminus f(\mathbb{B}^n) \neq \emptyset$, see [PC], Remark 1.5.

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2. PROOF OF THEOREM 1.6

We first recall the classical (conformal) modulus for path families in \mathbb{R}^n . Let Γ be a family of paths γ , i.e., continuous functions $\gamma : I \rightarrow \mathbb{R}^n$, where $I = [a, b]$ or $[a, b)$. We say that a Borel measurable function $\rho : \mathbb{R}^n \rightarrow [0, +\infty]$ is **admissible** for Γ if

$$\int_{\gamma} \rho ds \geq 1 \quad \forall \gamma \in \Gamma.$$

Then the **modulus** of Γ is

$$\text{Mod } \Gamma := \inf \left\{ \int_{\mathbb{R}^n} \rho(x)^n dx : \rho \text{ admissible} \right\}.$$

We recall two classical results concerning conformal modulus.

Lemma 2.1 (Poletsky's inequality, [Ric93], II Theorem 8.1). *Let $f : \Omega \rightarrow \mathbb{R}^n$ be a non-constant K -quasiregular mapping, and Γ a family of paths in Ω . Then*

$$\text{Mod } f\Gamma \leq K^{n-1} \text{Mod } \Gamma.$$

Lemma 2.2 ([Väi71], Theorem 10.12). *Suppose that J is a measurable set of radii, and $p \in \mathbb{R}^n$. For each $r \in J$, consider distinct points a_r, b_r in $\mathbb{S}^{n-1}(p, r)$. Set*

$$\Gamma = \{\gamma : [a, b] \rightarrow \mathbb{S}^{n-1}(p, r) \mid r \in J, \gamma \text{ connects } a_r \text{ and } b_r\}.$$

Then

$$\text{Mod } \Gamma \geq c_n \int_J \frac{dr}{r},$$

where $c_n > 0$ only depends on n .

Let f satisfy the assumptions of Theorem 1.6. We lose no generality by assuming $M = 1$. Let δ denote the largest radius so that

$$f(\mathbb{B}^n(0, \delta)) \subset \mathbb{B}^n.$$

In order to prove Theorem 1.6 we need to show that $\delta \geq C(n, K)$. Also, we let

$$\begin{aligned} F_0 &= \mathbb{B}^n \setminus f(\mathbb{B}^n), \\ F_1 &= \{y \in \mathbb{B}^n : \text{card } f^{-1}(y) = 1\}, \\ F_m &= \{y \in \mathbb{B}^n : \text{card } f^{-1}(y) \geq 2\} = \mathbb{B}^n \setminus (F_0 \cup F_1). \end{aligned}$$

By (1.4) and a change of variables, we have

$$\alpha_n > \int_{\{x \in \mathbb{B}^n : f(x) \in \mathbb{B}^n\}} J(x, f) dx = \int_{\mathbb{B}^n} \text{card } f^{-1}(y) dy.$$

Therefore $F_0 \neq \emptyset$.

We first prove Theorem 1.6 under the assumption

$$(2.1) \quad |F_0| \geq \alpha_n 100^{-n}.$$

We denote by T the set of those radii $0 < r < 1$ for which

$$\mathbb{S}^{n-1}(0, r) \cap F_0 \neq \emptyset.$$

Lemma 2.3. *Assume that (2.1) holds true. Then*

$$\int_T \frac{dr}{r} \geq n^{-1} 100^{-n}.$$

Proof. Since $r < 1$, we have

$$\begin{aligned} \int_T \frac{dr}{r} &= \omega_{n-1}^{-1} \int_T \int_{\mathbb{S}^{n-1}(0, r)} r^{-n} d\mathcal{H}_{n-1} dr \geq \omega_{n-1}^{-1} \int_{\mathbb{R}^n} \chi_{\{y : |y| \in T\}}(x) dx \\ &= \omega_{n-1}^{-1} |\{y : |y| \in T\}| \geq \omega_{n-1}^{-1} |F_0| \geq \alpha_n \omega_{n-1}^{-1} 100^{-n} = n^{-1} 100^{-n}. \end{aligned}$$

□

Proposition 2.4. *Theorem 1.6 holds true under assumption (2.1).*

Proof. By definition of T , for each $r \in T$, we can choose points $q_r \in F_0 \cap \mathbb{S}^{n-1}(0, r)$. Also, since $\overline{f(\mathbb{B}^n(0, \delta))}$ is a connected set containing 0 and a point in \mathbb{S}^{n-1} , for each $r \in T$, we can choose points $a_r \in \mathbb{B}^n(0, \delta)$ such that $f(a_r) \in \mathbb{S}^{n-1}(0, r)$. Then, for every path γ starting at $f(a_r)$ and joining $f(a_r)$ to q_r in $\mathbb{S}^{n-1}(0, r)$, every maximal lift γ' of γ starting at a_r accumulates on \mathbb{S}^{n-1} (see [Ric93], II.3 for the definition of a maximal lift). Hence, if we denote the family of all such lifts, for any $r \in T$, by Γ , we have

$$(2.2) \quad \text{Mod } \Gamma \leq \omega_{n-1} (\log \delta^{-1})^{1-n}.$$

On the other hand, by Lemmas 2.2 and 2.3,

$$(2.3) \quad \text{Mod } f\Gamma \geq c_n \int_T \frac{dr}{r} \geq c_n n^{-1} 100^{-n}.$$

By combining (2.2), (2.3) and Lemma 2.1, we have

$$c_n n^{-1} 100^{-n} \leq K^{n-1} \omega_{n-1} (\log \delta^{-1})^{1-n},$$

Thus Theorem 1.6 holds in this case with

$$r_0(n, K) = \exp \left(- \left(100^n c_n^{-1} n K^{n-1} \omega_{n-1} \right)^{\frac{1}{n-1}} \right).$$

□

We now treat the case when (2.1) fails. First we establish a geometric lemma.

Lemma 2.5. *Fix $q \in F_0$. Then there exists a point $w \in \mathbb{B}^n$, and $1/4 \leq s < 1$, such that for all $r \in (s, \sqrt{3}s)$, we have $q \in \mathbb{B}^n(w, r)$ and $\mathbb{S}^{n-1}(w, r) \cap f(\mathbb{B}^n(0, \delta)) \neq \emptyset$.*

Proof. First assume $|q| \leq 1/2$. Then, since $\overline{f(\mathbb{B}^n(0, \delta))}$ is a connected set containing 0 and a point in \mathbb{S}^{n-1} ,

$$\mathbb{S}^{n-1}(0, r) \cap f(\mathbb{B}^n(0, \delta)) \neq \emptyset \quad \forall r \in \left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right).$$

Hence we may choose $w = 0$, $s = 1/2$.

Thus assume $|q| > 1/2$. Choose $p \in \mathbb{B}^n(0, \delta)$ such that $|f(p)| = |q|$. Consider the triangle with vertices 0, $f(p)$ and $q/2$. Then, if the angle of the triangle at $q/2$ is less than $\pi/2$, we have, for each $r \in (|q|/2, \sqrt{3}|q|/2)$,

$$0, q \in \mathbb{B}^n(q/2, r), \quad f(p) \notin \mathbb{B}^n(q/2, r).$$

Since $0, f(p) \in f(\mathbb{B}^n(0, \delta))$, there exists, for each such r , a point $\eta_r \in \mathbb{B}^n(0, \delta)$ such that $f(\eta_r) \in \mathbb{S}^{n-1}(q/2, r)$. Hence we may choose $w = q/2$ and $s = |q|/2 > 1/4$ in this case.

If the angle is greater than or equal to $\pi/2$, we have, for each $r \in (|q|/2, \sqrt{3}|q|/2)$,

$$f(p), q \in \mathbb{B}^n \left(\frac{f(p) + q}{2}, r \right), \quad 0 \notin \mathbb{B}^n \left(\frac{f(p) + q}{2}, r \right).$$

Hence we may in this case choose $w = (f(p) + q)/2$ and $s = |q|/2$. □

Let q , w , and s be as in Lemma 2.5. We denote by G the set of all radii $r \in (s, \sqrt{3}s)$ for which

$$F_1 \cap \mathbb{S}^{n-1}(w, r) \neq \emptyset.$$

Lemma 2.6. *If (2.1) fails, then*

$$\int_G \frac{dr}{r} \geq n^{-1} 100^{-n}.$$

Proof. As in Lemma 2.3, we have

$$(2.4) \quad \int_G \frac{dr}{r} \geq \omega_{n-1}^{-1} \int_G \int_{\mathbb{S}^{n-1}(w, r)} d\mathcal{H}_{n-1} r^{-n} dr \geq \omega_{n-1}^{-1} |F_1 \cap (\mathbb{B}^n(w, \sqrt{3}s) \setminus \overline{\mathbb{B}^n}(w, s))|.$$

By our assumption (1.4) and a change of variables,

$$|F_1| + 2|F_m| \leq \int_{\mathbb{B}^n} \text{card } f^{-1}(y) dy = \int_{\{x \in \mathbb{B}^n : f(x) \in \mathbb{B}^n\}} J(x, f) dx < \alpha_n = |F_0| + |F_1| + |F_m|.$$

So

$$(2.5) \quad |F_m| \leq |F_0| \leq \alpha_n 100^{-n},$$

where the last inequality holds true since we assume the converse of (2.1).

On the other hand, since $w \in \mathbb{B}^n$ and $s \geq 1/4$, we have

$$(2.6) \quad |(\mathbb{B}^n(w, \sqrt{3}s) \setminus \overline{\mathbb{B}^n}(w, s)) \cap \mathbb{B}^n| \geq \alpha_n 10^{-n},$$

and combining (2.5) and (2.6) yields

$$(2.7) \quad \begin{aligned} |F_1 \cap (\mathbb{B}^n(w, \sqrt{3}s) \setminus \overline{\mathbb{B}^n}(w, s))| &= |(\mathbb{B}^n(w, \sqrt{3}s) \setminus \overline{\mathbb{B}^n}(w, s)) \cap \mathbb{B}^n| \\ &- |(\mathbb{B}^n(w, \sqrt{3}s) \setminus \overline{\mathbb{B}^n}(w, s)) \cap (F_0 \cup F_m)| \geq \alpha_n 100^{-n}. \end{aligned}$$

The Lemma follows by combining (2.4) and (2.7). \square

For each $r \in G$, choose points $p_r \in f^{-1}(F_1)$, $a_r \in \mathbb{B}^n(0, \delta)$ such that

$$f(p_r), f(a_r) \in \mathbb{S}^{n-1}(w, r).$$

Denote

$$\begin{aligned} G_1 &= \{r \in G : |p_r| \geq \delta^{\frac{1}{2}}\}, \\ G_2 &= \{r \in G : |p_r| < \delta^{\frac{1}{2}}\} = G \setminus G_1. \end{aligned}$$

Then, by Lemma 2.6, either (2.1) holds, or else we have one of

$$(2.8) \quad \int_{G_1} \frac{dr}{r} \geq 2^{-1} n^{-1} 100^{-n},$$

or

$$(2.9) \quad \int_{G_2} \frac{dr}{r} \geq 2^{-1} n^{-1} 100^{-n}.$$

Proposition 2.7. *Theorem 1.6 holds true under assumption (2.8).*

Proof. For each $r \in G_1$ and each γ starting at $f(a_r)$ and joining $f(a_r)$ to $f(p_r)$ in $\mathbb{S}^{n-1}(w, r)$, consider a maximal lift γ' of γ starting at a_r . Then, since $\text{card } f^{-1}(f(p_r)) = 1$, either γ' accumulates to \mathbb{S}^{n-1} , or γ' ends at p_r ; in any case, γ' starts at $\mathbb{B}^n(0, \delta)$ and leaves $\mathbb{B}^n(0, \delta^{\frac{1}{2}})$.

Denote the family of all such γ' by Γ . Then we have

$$(2.10) \quad \text{Mod } \Gamma \leq \omega_{n-1} \left(\log \frac{\delta^{\frac{1}{2}}}{\delta} \right)^{1-n} = \omega_{n-1} \left(\log \delta^{-\frac{1}{2}} \right)^{1-n}.$$

On the other hand, combining Lemma 2.2 and (2.8) yields

$$(2.11) \quad \text{Mod } f\Gamma \geq c_n 2^{-1} n^{-1} 100^{-n}.$$

Furthermore, combining (2.10), (2.11) and Lemma 2.1 gives

$$c_n 2^{-1} n^{-1} 100^{-n} \leq K^{n-1} \omega_{n-1} \left(\log \delta^{\frac{-1}{2}} \right)^{1-n},$$

Thus Theorem 1.6 holds in this case with

$$r_0(n, K) = \exp \left(-2 \left(100^n 2 c_n^{-1} n K^{n-1} \omega_{n-1} \right)^{\frac{1}{n-1}} \right).$$

□

In order to finish the proof of Theorem 1.6, we need the following auxiliary result.

Lemma 2.8. *For each $r \in G_2$ there exists $\tau_r \in \mathbb{S}^{n-1}(0, \delta^{\frac{1}{4}})$ such that $f(\tau_r) \in \mathbb{B}^n(w, r)$.*

Proof. Let U_r be any component of $f^{-1}(\mathbb{B}^n(w, r))$ intersecting $\mathbb{B}^n(0, \delta)$. Such a component exists by Lemma 2.5. Also, by Lemma 2.5, $\mathbb{B}^n(w, r) \setminus f(\mathbb{B}^n) \neq \emptyset$, and hence $f|_{U_r} : U_r \rightarrow \mathbb{B}^n(w, r)$ is not onto. Thus, by [Ric93], I Lemma 4.7,

$$\mathbb{S}^{n-1}(0, t) \cap U_r \neq \emptyset \quad \forall t \in (\delta, 1).$$

Choose $k_r \in U_r \cap \mathbb{S}^{n-1}(0, \delta^{\frac{1}{4}})$, and consider all paths joining k_r to $-k_r$ in $\mathbb{S}^{n-1}(0, \delta^{\frac{1}{4}})$. If none of the images of these paths intersects $\mathbb{S}^{n-1}(w, r)$, we have

$$(2.12) \quad f(\mathbb{S}^{n-1}(0, \delta^{\frac{1}{4}})) \subset \mathbb{B}^n(w, r).$$

Since f is open,

$$\partial f(\mathbb{B}^n(0, \delta^{\frac{1}{4}})) \subset f(\mathbb{S}^{n-1}(0, \delta^{\frac{1}{4}})),$$

and since $f(\mathbb{B}^n(0, \delta^{\frac{1}{4}}))$ is bounded, (2.12) further implies

$$(2.13) \quad f(\mathbb{B}^n(0, \delta^{\frac{1}{4}})) \subset \mathbb{B}^n(w, r).$$

By Lemma 2.5 there are, however, points $x \in \mathbb{B}^n(0, \delta)$ such that $f(x) \notin \mathbb{B}^n(w, r)$ which contradicts (2.13). The proof is complete. □

Proposition 2.9. *Theorem 1.6 holds true under assumption (2.9).*

Proof. For each $r \in G_2$, and each γ starting at $f(\tau_r)$ (where τ_r is as in Lemma 2.8) and joining $f(\tau_r)$ to $f(p_r)$ in $\mathbb{S}^{n-1}(w, r)$, consider a maximal lift γ' of γ starting at τ_r . Then, since $\text{card } f^{-1}(f(p_r)) = 1$, either γ' accumulates to \mathbb{S}^{n-1} , or γ' ends at p_r . We denote the family of all such γ' for which the first case occurs by Γ_1 , the family of all γ' for which the second case occurs by Γ_2 , and $\Gamma = \Gamma_1 \cup \Gamma_2$.

Then, since each $\gamma' \in \Gamma_1$ connects $\mathbb{S}^{n-1}(0, \delta^{\frac{1}{4}})$ to \mathbb{S}^{n-1} ,

$$(2.14) \quad \text{Mod } \Gamma_1 \leq \omega_{n-1} \left(\log \delta^{\frac{-1}{4}} \right)^{1-n}.$$

Similarly, since $p_r \in \mathbb{B}^n(0, \delta^{\frac{1}{2}})$ for all $r \in G_2$,

$$(2.15) \quad \text{Mod } \Gamma_2 \leq \omega_{n-1} \left(\log \frac{\delta^{\frac{1}{4}}}{\delta^{\frac{1}{2}}} \right)^{1-n} = \omega_{n-1} \left(\log \delta^{\frac{-1}{4}} \right)^{1-n}.$$

By Lemma 2.2 and (2.9),

$$(2.16) \quad \text{Mod } f\Gamma \geq c_n 2^{-1} n^{-1} 100^{-n}.$$

Hence, combining (2.14), (2.15), (2.16) and Lemma 2.1 yields

$$\begin{aligned} c_n 2^{-1} n^{-1} 100^{-n} &\leq \text{Mod } f\Gamma \leq K^{n-1} \text{Mod } \Gamma \leq K^{n-1} (\text{Mod } \Gamma_1 + \text{Mod } \Gamma_2) \\ &\leq 2K^{n-1} \omega_{n-1} \left(\log \delta^{\frac{-1}{4}} \right)^{1-n}, \end{aligned}$$

Thus Theorem 1.6 holds in this case with

$$r_0(n, K) = \exp \left(-4 \left(100^n 4c_n^{-1} n K^{n-1} \omega_{n-1} \right)^{\frac{1}{n-1}} \right).$$

□

The proof of Theorem 1.6 follows by combining Propositions 2.4, 2.7 and 2.9.

3. BEURLING'S MODULUS ESTIMATE

Suppose f is K -quasiregular in a neighborhood of the closed unit ball $\overline{\mathbb{B}}^n$, and for some fixed $0 < r < 1$ let $M := \max_{r\overline{\mathbb{B}}^n} |f|$. Recall that for $s > M$ we define $F_s^* = \{\zeta \in \mathbb{S}^{n-1} : |f(\zeta)| \geq s\}$ and for $M < t < s$ we have $E_t = \{x \in \mathbb{B}^n : |f(x)| = t\}$. Consider the family Γ_s consisting of the paths in \mathbb{B}^n starting at $r\overline{\mathbb{B}}^n$ and ending at F_s^* . We claim that

$$(3.1) \quad \text{Mod } \Gamma_s \leq K \left(\int_M^s \frac{dt}{(\mathcal{A}_{n-1}f(E_t))^{\frac{1}{n-1}}} \right)^{1-n}.$$

Recall that $\mathcal{A}_{n-1}f(E_t) = \int_{\mathbb{S}^{n-1}(0,t)} \text{card } f^{-1}(y) d\mathcal{H}_{n-1}(y)$.

Proof. Set $\rho : \mathbb{R}^n \rightarrow [0, \infty)$,

$$\rho(x) = \left(\int_M^s \frac{du}{(\mathcal{A}_{n-1}f(E_u))^{\frac{1}{n-1}}} \right)^{-1} \frac{\|Df(x)\|}{(\mathcal{A}_{n-1}f(E_t))^{\frac{1}{n-1}}} \quad \text{when } |f(x)| = t \in (M, s),$$

and $\rho(x) = 0$ otherwise. Then, for each $\gamma \in \Gamma_s$,

$$\int_\gamma \rho ds \geq \left(\int_M^s \frac{du}{(\mathcal{A}_{n-1}f(E_u))^{\frac{1}{n-1}}} \right)^{-1} \int_{f(\gamma)} (\mathcal{A}_{n-1}f(E_{|\cdot|}))^{\frac{-1}{n-1}} ds \geq 1.$$

Moreover, if we denote

$$I(M, s) = \int_M^s \frac{du}{(\mathcal{A}_{n-1}f(E_u))^{\frac{1}{n-1}}}$$

and

$$A(M, s) = f^{-1}(\mathbb{B}^n(0, s) \setminus \overline{\mathbb{B}^n(0, M)}),$$

we have

$$\begin{aligned}
\text{Mod } \Gamma_s &\leq \int_{\mathbb{R}^n} \rho(x)^n dx = I(M, s)^{-n} \int_{A(M, s)} \frac{\|Df(x)\|^n}{(\mathcal{A}_{n-1}f(E_{|f(x)|}))^{\frac{n}{n-1}}} dx \\
&\leq KI(M, s)^{-n} \int_{A(M, s)} \frac{J(x, f)}{(\mathcal{A}_{n-1}f(E_{|f(x)|}))^{\frac{n}{n-1}}} dx \\
&= KI(M, s)^{-n} \int_{f(A(M, s))} \frac{\text{card } f^{-1}(y)}{(\mathcal{A}_{n-1}f(E_{|y|}))^{\frac{n}{n-1}}} dy \\
&= KI(M, s)^{-n} \int_M^s (\mathcal{A}_{n-1}f(E_t))^{\frac{-n}{n-1}} \int_{\mathbb{S}^{n-1}(0, t)} \text{card } f^{-1}(\varphi) d\mathcal{H}_{n-1}(\varphi) dt = KI(M, S)^{1-n}.
\end{aligned}$$

□

4. CAPACITY AND SYMMETRIZATION

We recall that a **condenser** is a pair (Ω, K) with $\Omega \subset \mathbb{R}^n$, Ω open and K compact with $\emptyset \neq K \subset \Omega$. Also, the **conformal capacity** of (Ω, K) is

$$\text{Cap}(\Omega, K) := \inf\{\|\nabla u\|_{L^n(\Omega)}^n : u \in W_0^{1,p}(\Omega), u|_V \geq 1, \text{ for some } V \text{ open, } V \supset K\}$$

where $W_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ (the smooth functions compactly supported in Ω) in the norm

$$\|u\|_{W_0^{1,p}(\Omega)} = \left(\int_{\Omega} |u(x)|^n + |\nabla u(x)|^n dx \right)^{\frac{1}{n}}.$$

By Proposition II.10.2 of [Ric93], if $\Gamma(\Omega, K)$ is the family of all paths $\gamma : [a, b] \rightarrow \Omega$ such that $\gamma(a) \in K$ and $\lim_{t \rightarrow b} \gamma(t) \in \partial\Omega$, then

$$(4.1) \quad \text{Cap}(\Omega, K) = \text{Mod } \Gamma(\Omega, K).$$

We are mainly interested in measuring the sets F_s^* defined in Section 3, which are compact subsets of \mathbb{S}^{n-1} . Therefore, we will fix $0 < r < 1$ to be determined later, consider the spherical ring $A(r) = \{x \in \mathbb{R}^n : r < |x| < 1/r\}$, $0 < r < 1$, and compute $\text{Cap}(A(r), F_s^*)$.

By the symmetry rule, cf. [GM05] IV(3.4), if $F \subset \mathbb{S}^{n-1}$, we have:

$$(4.2) \quad \text{Mod } \Gamma_s = \frac{1}{2} \text{Mod } \Gamma(A(r), F) = \frac{1}{2} \text{Cap}(A(r), F).$$

Also, if $F \subset \mathbb{S}^{n-1}$, let $\mathcal{C}(F)$ be the spherical cap centered at $e_1 = (1, 0, \dots, 0)$ with $\mathcal{H}_{n-1}(\mathcal{C}(F)) = \mathcal{H}_{n-1}(F)$. By spherical symmetrization, see [Geh61],

$$(4.3) \quad \text{Cap}(A(r), \mathcal{C}(F)) \leq \text{Cap}(A(r), F).$$

By [Geh61], Theorem 4, we see that, when $\mathcal{H}_{n-1}(F) \leq \epsilon(r, n)$,

$$(4.4) \quad \text{Cap}(A(r), \mathcal{C}(F)) \geq \omega_{n-1} \log^{1-n} \frac{C_2}{\mathcal{H}_{n-1}(F)^{\frac{1}{n-1}}},$$

where $C_2 > 0$ depends only on r and n (the results in [Geh61] are stated for $n = 3$ only, but they extend to all dimensions).

Putting (3.1), (4.1), (4.2), (4.3), and (4.4) together, we obtain (1.3) and thus we have proved Theorem 1.5 for $\mathcal{H}_{n-1}(F_s^*) \leq \epsilon(r, n)$. If $\mathcal{H}_{n-1}(F_s^*) > \epsilon(r, n)$, then the arguments above show that

$$(4.5) \quad \text{Mod } \Gamma_s \geq C(r, n).$$

Combining (4.5) with (3.1) yields

$$\int_M \frac{dt}{(\mathcal{A}_{n-1}f(E_t))^{\frac{1}{n-1}}} \leq C(r, n, K).$$

Hence increasing C_1 if necessary gives Theorem 1.5 for all $s > M$.

We finish this section by briefly commenting on the real-valued case mentioned in the introduction. Suppose that $u : \mathbb{B}^n \rightarrow \overline{\mathbb{R}}$ belongs to $W^{1,n}(\mathbb{B}^n)$ and satisfies

$$(4.6) \quad \int_{\frac{1}{2}\mathbb{B}^n} u(x) dx = 0.$$

Then, by the Poincaré inequality and (4.6),

$$|A_T| = |\{x \in \frac{1}{2}\mathbb{B}^n : |u| \leq T\}| \geq C(n)$$

for large enough T depending only on n and the Sobolev norm of u . Hence, by applying arguments similar to the ones above to the n -capacity related to the sets A_T and $U_s^* = \{y \in \mathbb{S}^{n-1} : |u^*(y)| \geq s\}$, we have an estimate for the \mathcal{H}_{n-1} -measure of U_s^* in terms of s , T and $\int_{\{x \in \mathbb{B}^n : |u| \leq s\}} |\nabla u(x)|^n dx$.

5. EXPONENTIAL INTEGRABILITY

In this section we prove Theorem 1.1 by using the results established in previous sections and arguments similar to those used in [Mar89]. Let f be a K -quasiregular mapping defined in a neighborhood of \mathbb{B}^n and satisfying $f(0) = 0$ and (1.2). We denote

$$\beta = (n-1) \left(\frac{n}{2K} \right)^{\frac{1}{n-1}}.$$

Then

$$\alpha_n^{\frac{1}{n-1}} \beta = (n-1) \left(\frac{\omega_{n-1}}{2K} \right)^{\frac{1}{n-1}}.$$

Notice that we lose no generality by assuming that f is defined in a neighborhood of \mathbb{B}^n : if we consider a sequence (r_j) increasing to one, and functions f_j , $f_j(x) = f(r_j x)$, then the existence of radial limits at almost every $\varphi \in \mathbb{S}^{n-1}$ and Fatou's lemma yield

$$\int_{\mathbb{S}^{n-1}} \exp \left(\beta |f^*(\zeta)|^{\frac{n}{n-1}} \right) d\mathcal{H}_{n-1}(\zeta) \leq \liminf_j \int_{\mathbb{S}^{n-1}} \exp \left(\beta |f_j^*(\zeta)|^{\frac{n}{n-1}} \right) d\mathcal{H}_{n-1}(\zeta).$$

By the Cavalieri principle,

$$(5.1) \quad \int_{\mathbb{S}^{n-1}} \exp \left(\beta |f(\zeta)|^{\frac{n}{n-1}} \right) d\mathcal{H}_{n-1}(\zeta) = \omega_{n-1} + \frac{\beta n}{n-1} \int_0^\infty s^{\frac{1}{n-1}} \mathcal{H}_{n-1}(F_s^*) \exp(\beta s^{\frac{n}{n-1}}) ds.$$

We choose $r_0 = r_0(n, K)$ as in Theorem 1.6, and let $M = \max_{|x| \leq r_0} |f(x)|$. Note that by Corollary 1.7 and (1.2), we have $M < 1$ and

$$(5.2) \quad \int_{\{x \in \mathbb{B}^n : f(x) \in \mathbb{B}^n(0, M)\}} J(x, f) dx = \int_0^M \mathcal{A}_{n-1} f(E_t) dt \geq \alpha_n M^n.$$

Using (1.3) and (5.1), we are reduced to estimate

$$(5.3) \quad \int_0^{\|f\|_\infty} s^{\frac{1}{n-1}} \exp(\beta s^{\frac{n}{n-1}} - \psi(s)) ds,$$

where $\psi(s) = 0$ for $0 < s \leq M$ and

$$\psi(s) = \alpha_n^{\frac{1}{n-1}} \beta \int_M^s \frac{dt}{(\mathcal{A}_{n-1} f(E_t))^{\frac{1}{n-1}}}$$

for $s \geq M$. We modify ψ as follows: for $0 < s \leq M$, set

$$\tilde{\psi}(s) := \mu s,$$

and for $s \geq M$,

$$\tilde{\psi}(s) := \psi(s) + \mu M,$$

where

$$\mu = \left(\frac{M \beta^{n-1} \alpha_n}{\int_0^M \mathcal{A}_{n-1} f(E_t) dt} \right)^{\frac{1}{n-1}}.$$

Note that $\tilde{\psi}$ is strictly increasing for $0 < s \leq \|f\|_\infty$ and constant, equal to $\|\tilde{\psi}\|_\infty$, for $s > \|f\|_\infty$. Also $\tilde{\psi}(0) = 0$. Finally, $\tilde{\psi} \leq \psi + \mu M$. So, by (5.2), and since $M < 1$, it is enough to estimate (5.3) with ψ replaced by $\tilde{\psi}$.

Let $\phi(y) := \tilde{\psi}^{-1}(y)$ for $0 < y \leq \|\tilde{\psi}\|_\infty$ and $\phi(y) := \|f\|_\infty$ for $y > \|\tilde{\psi}\|_\infty$, so that ϕ is strictly increasing for $0 < y \leq \|\tilde{\psi}\|_\infty$ and $\phi(0) = 0$.

Changing variables $y = \tilde{\psi}(s)$ the integral (5.3) becomes

$$\int_0^{\|\tilde{\psi}\|_\infty} \exp(\beta \phi(y)^{\frac{n}{n-1}} - y) \phi'(y) \phi(y)^{\frac{1}{n-1}} dy$$

which, since $\phi' \geq 0$, is less than or equal to the same integral but from 0 to ∞ . Integrating by parts we then need to estimate

$$(5.4) \quad \int_0^\infty \exp(\beta \phi(y)^{\frac{n}{n-1}} - y) dy = \int_0^\infty \exp((\beta^{\frac{n-1}{n}} \phi(y))^{\frac{n}{n-1}} - y) dy.$$

We have

$$\beta^{\frac{n-1}{n}} \phi'(y) = \begin{cases} \beta^{\frac{n-1}{n}} \mu^{-1}, & 0 < y < \mu M, \\ \alpha_n^{\frac{-1}{n-1}} \beta^{\frac{-1}{n}} (\mathcal{A}_{n-1} f(E_{\phi(y)}))^{\frac{1}{n-1}}, & \mu M < y < \|\tilde{\psi}\|_\infty. \end{cases}$$

Thus, by changing variables with $s = \phi(y)$, and by our choice of μ ,

$$\begin{aligned} \int_0^\infty (\beta^{\frac{n-1}{n}} \phi'(y))^n dy &= \int_0^{\mu M} \beta^{n-1} \mu^{-n} dy + \alpha_n^{\frac{-n}{n-1}} \beta^{-1} \int_{\mu M}^{\|\tilde{\psi}\|_\infty} (\mathcal{A}_{n-1} f(E_{\phi(y)}))^{\frac{n}{n-1}} dy \\ &= \beta^{n-1} M \mu^{1-n} + \alpha_n^{-1} \int_M^{\|f\|_\infty} \mathcal{A}_{n-1} f(E_t) dt \\ &\leq \alpha_n^{-1} \int_0^\infty \mathcal{A}_{n-1} f(E_t) dt \leq 1. \end{aligned}$$

By applying equation (6), page 1080 of [Mos71] to $\beta^{\frac{n-1}{n}} \phi$, we conclude that (5.4) is bounded from above by a constant depending only on n . The proof of Theorem 1.1 is complete.

We finally note that, under the assumptions of Theorem 1.1, the left hand side of (5.1) is finite for every $\beta > 0$. We fix $M > 0$, to be chosen later. After applying Theorem 1.5 to the right hand term in (5.1), we need to show that

$$(5.5) \quad \int_M^\infty s^{\frac{1}{n-1}} \exp \left(\beta s^{\frac{n}{n-1}} - C \int_M^s \frac{dt}{(\mathcal{A}_{n-1} f(E_t))^{\frac{1}{n-1}}} \right) ds$$

is finite, where $C > 0$.

By Hölder's inequality,

$$(5.6) \quad s - M = \int_M^s \frac{(\mathcal{A}_{n-1} f(E_t))^{\frac{1}{n}}}{(\mathcal{A}_{n-1} f(E_t))^{\frac{1}{n}}} dt \leq \left(\int_M^s \frac{dt}{(\mathcal{A}_{n-1} f(E_t))^{\frac{1}{n-1}}} \right)^{\frac{n-1}{n}} \left(\int_M^s \mathcal{A}_{n-1} f(E_t) dt \right)^{\frac{1}{n}}.$$

By our assumption $\int_0^\infty \mathcal{A}_{n-1} f(E_t) dt$ is finite. Thus, by choosing M large enough so that

$$\left(\int_M^\infty \mathcal{A}_{n-1} f(E_t) dt \right)^{\frac{-1}{n-1}} > \frac{2\beta}{C},$$

and combining this with (5.6), we can estimate (5.5) from above by

$$\int_M^\infty s^{\frac{1}{n-1}} \exp(\beta(s^{\frac{n}{n-1}} - 2(s-M)^{\frac{n}{n-1}})) ds,$$

which is clearly finite.

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